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Invariance quantum groups of the deformed oscillator algebra

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Abstract. A differential calculus is set up on a deformation of the oscillator algebra. It is uniquely determined by the requirement of invariance under a seven-dimensional quantum group. The quantum space and its associated differential calculus are also shown to be invariant under a nine generator quantum group containing the previous one.

1. Introduction

Several kinds of differential calculi have been introduced concerning quantum groups and quantum spaces and have been widely studied [1–3]. However, there is one important case that deserves special treatment because of its importance in physics, namely the case of the oscillator (or Weyl–Heisenberg) algebra. Indeed, in its usual form it represents the algebra of observables in quantum mechanics. After deformation, it is still an algebra of observables but for a different quantization.

The problem of constructing a differential calculus on such an algebra can be tackled in different ways. Here we favour the invariance approach. Given the quantum space of variables (or observables), we first construct the set of seven-element quantum matrices preserving that space (section 2). The space of differentials is then determined in two independent ways: either by postulating the existence of a R-matrix (section 3) or by constructing directly a new invariant space (section 4). The uniqueness of the result emphasizes the power of the approach based on covariance. Finally, in section 5, the quantum group preserving simultaneously the spaces of variables and differentials is explicitly described. Moreover, a larger quantum group with nine generators is also shown to preserve the same spaces. Both groups can be endowed with a structure of Hopf algebra.

2. The quantum space and its invariance algebra

The Weyl–Heisenberg algebra written in homogeneous form is the free associative algebra generated by three operators x^i satisfying the following quadratic relations:

$$(R)\begin{cases} x^1x^2 - x^2x^1 - s(x^3)^2 = 0 \\ x^1x^3 = x^3x^1 \\ x^2x^3 = x^3x^2. \end{cases} \tag{1}$$

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This algebra will be denoted by $\mathcal{C}\langle x\rangle/R$. It is invariant under the seven-parameter Lie subgroup G of $GL(3)$ consisting of matrices T such that $T_1^3 = T_2^3 = 0$ and $T_1^1T_2^2 - T_2^1T_1^2 = (T_3^3)^2$.

Deforming this algebra, we consider the quantum space obtained when relations (1) are replaced by

$$(R_{xx}) \begin{cases} x^1x^2 - qx^2x^1 - s(x^3)^2 = 0 \\ x^1x^3 - ux^3x^1 = 0 \\ x^2x^3 - u^{-1}x^3x^2 = 0. \end{cases} \tag{2}$$

This is the most general deformation of relations (1) which preserves their quadratic character and does not introduce any new term. The resulting quantum space contains in particular the q-oscillator which corresponds to the choice $q = u^{-2}$, $s = 1$ when $x^1 = a$, $x^2 = a^\dagger$ and $x^3 = q^{-N/2}$.

Let V be the vector space of column vectors X with elements x^i. The matrix T now has non-commuting elements and can be made to act on X according to the usual techniques developed in [5] and [6]:

$$\delta\colon X \longrightarrow \delta(X) = T \otimes X. \tag{3}$$

Relations R_{xx} defined by (2) will be preserved by this action provided the following commutation relations hold

$$\begin{aligned}
&T_1^1T_3^3 = T_3^3T_1^1 \qquad T_1^2T_3^3 = u^{-2}T_3^3T_1^2 \\
&T_2^1T_3^3 = u^2T_3^3T_2^1 \qquad T_2^2T_3^3 = T_3^3T_2^2 \\
&T_3^1T_3^3 = uT_3^3T_3^1 \qquad T_3^2T_3^3 = u^{-1}T_3^3T_3^2 \\
&T_1^1T_1^2 = qT_1^2T_1^1 \qquad T_2^1T_2^2 = qT_2^2T_2^1 &&(4)\\
&uT_1^1T_3^2 - qT_3^2T_1^1 = quT_1^2T_3^1 - T_3^1T_1^2 &&(5)\\
&T_2^1T_3^2 - quT_3^2T_2^1 = qT_2^2T_3^1 - uT_3^1T_2^2 &&(6)\\
&T_1^1T_2^2 - T_2^2T_1^1 = qT_1^2T_2^1 - q^{-1}T_2^1T_1^2 &&(7)\\
&(T_1^1T_2^2 - qT_1^2T_2^1)s = s(T_3^3)^2 - T_3^1T_3^2 + qT_3^2T_3^1. &&(8)
\end{aligned}$$

In that case, the action δ becomes defined as a mapping of $\mathbb{C}\langle x\rangle/R_{xx}$ onto itself.

3. *R*-matrix and invariant differential calculus

In the present case, a differential calculus on $\mathbb{C}\langle x\rangle/R_{xx}$ that is invariant under the action of T can be constructed if there exists a matrix $\widehat{R}$ with the following properties [5]:

— $\widehat{R}$ is defined by the relations

$$\widehat{R}^{ji}_{kl}T^k_mT^l_n = T^j_lT^i_k\widehat{R}^{lk}_{mn}. \tag{9}$$

— $\widehat{R}$ has two eigenspaces, V_1 and V_2 that can be identified with the variables and the one-forms quantum spaces, respectively. Space V_1 has dimension six and is determined by the relations (R_{xx}) given in (2).

The determination of $\widehat{R}$ is performed by assuming that relations (4)–(8) can be cast into the form (9) and solving the corresponding equations. In addition, we impose two natural requirements.

(i) The determinant of T is different from zero so that the set of matrices T can be made into a Hopf algebra.

(ii) The ordering of monomials such as $T^i_j T^k_l T^m_n$ is independent of the procedure used, when the associativity of the algebra $C\langle T\rangle$ is taken into account.
With these conditions, it is found that a $\widehat{R}$-matrix exists only if $q = u^2$ and it is given by:

$$\widehat{R} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u^2 & 0 & 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 \\ 0 & u^{-2} & 0 & 0 & 0 & 0 & 0 & 0 & -s/u^2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/u & 0 \\ 0 & 0 & 1/u & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

It can be shown that this matrix $\widehat{R}$ is equal to its inverse and verifies the Yang–Baxter equation

$$(\widehat{R} \otimes 1)(1 \otimes \widehat{R})(\widehat{R} \otimes 1) = (1 \otimes \widehat{R})(\widehat{R} \otimes 1)(1 \otimes \widehat{R}) \tag{11}$$

where 1 is the unit matrix of $GL(3)$. The matrix $\widehat{R}$ has two eigenspaces that correspond to the variables quantum space defined by the relations R_{xx} and to the one-forms quantum space defined by

$$R_{\xi\xi}\colon \begin{cases} (\xi^1)^2 = 0 & (\xi^2)^2 = 0 \\ (\xi^3)^2 = 0 & \xi^2\xi^1 = -u^{-2}\xi^1\xi^2 \\ \xi^1\xi^3 = -u\xi^3\xi^1 & \xi^2\xi^3 = -u^{-1}\xi^3\xi^2. \end{cases} \tag{12}$$

Result 1. If the existence of a $\widehat{R}$-matrix is assumed and if conditions (i) and (ii) are satisfied, then an invariant differential calculus can be set up on $C\langle x\rangle/R_{xx}$ if and only if $q = u^2$.

In the next section, the same result is obtained without assuming the existence of an $\widehat{R}$-matrix.

4. Invariant exterior algebra

Consider a vector $\Xi \in V$ with components ξ^i, $i = 1, 2, 3$ and form $V \otimes V$. The space of invariant forms can be constructed directly as an invariant subspace V_2 of $V \otimes V$ that is supposed to be of dimension three or less. This will be done by finding the possible invariant relations between the $\xi^i\xi^j$.

An important tool for this derivation is the introduction of a degree d° on $C\langle T\rangle$, i.e. of a homomorphism from $C\langle T\rangle$ into Z. Explicitly, the degree operation d° associates with each element T^i_j the power of u present in the commutation relations (4) of that element with T^3_3. Assuming that $u \neq 1$, we find the degrees of all the elements of T

$$\begin{aligned} &d^\circ(T^1_1) = d^\circ(T^2_2) = d^\circ(T^3_3) = 0 \\ &d^\circ(T^1_2) = 2 \\ &d^\circ(T^2_1) = -2 \\ &d^\circ(T^1_3) = 1 \\ &d^\circ(T^2_3) = -1. \end{aligned} \tag{13}$$

The degrees of all the monomials in $C\langle T\rangle$ can then be deduced.

Consider a quadratic relation $R_1 = 0$ between the ξ. After action of the homomorphism $\delta\otimes\delta$ defined in (3), a new relation $(T\otimes T)\otimes R_1 = 0$ is obtained which contains monomials $T^i_j T^k_l$ of known degrees. The invariance of relation $R_1 = 0$ then implies that terms of different degrees vanish separately. Applying this remark systematically allows all possible invariant relations to be found as will now be sketched.

Let $\xi'^i\xi'^j$ denote the components of the vector $\Xi' \otimes \Xi' \equiv (T\otimes T)\otimes(\Xi\otimes\Xi)$. The monomial $(\xi'^1)^2$ is the only one containing a term of degree four. Hence it cannot be involved in any quadratic relation except $(\xi^1)^2 = 0$. The same reasoning with $(\xi'^2)^2$ and degree (-4) leads to $(\xi^2)^2 = 0$. In the remaining transformed relations, terms of degree $(+2)$ (respectively (-2)) come from monomials $\xi'^1\xi'^3$ and $\xi'^3\xi'^1$ (respectively $\xi'^2\xi'^3$ and $\xi'^3\xi'^2$). This implies that $\xi^1\xi^3$ and $\xi^2\xi^3$ are independent monomials. To generate V_2, we need only a third independent monomial which can be chosen as $\xi^1\xi^2, \xi^2\xi^1$ or $(\xi^3)^2$. Choosing $\xi^1\xi^2$, we can write the general form of the invariant relations as

$$\begin{aligned}
&(\xi^1)^2 = 0\\
&(\xi^2)^2 = 0\\
&(\xi^3)^2 = k\xi^1\xi^2\\
&\xi^2\xi^1 = \xi\xi^1\xi^2\\
&\xi^3\xi^1 = \lambda\xi^1\xi^3 + \lambda_{12}\xi^1\xi^2\\
&\xi^3\xi^2 = \mu\xi^2\xi^3 + \mu_{12}\xi^1\xi^2.
\end{aligned} \tag{14}$$

The conditions of invariance of these commutation relations yield new constraints on the elements of matrix T which are consistent provided that $k = \lambda_{12} = \mu_{12} = 0$.

If instead we choose $(\xi^3)^2$ as the third independent monomial, we obtain a different situation only if $\xi^1\xi^2 = 0$. In that case $\xi'^1\xi'^2$ contains only one term of degree 0, namely $T^1_3 T^2_3 (\xi^3)^2$ that cannot vanish. Hence this case is impossible.

We prove in the same way that the dimension of V_2 cannot be equal to two.

Thus, assuming that the differentials are related by six (or more) relations, we have shown that the only possible set is given by (14) with $k = \lambda_{12} = \mu_{12} = 0$. The constraint of invariance of the resulting relations has been explicitly studied in [7]. The consistency requirement for the ordering of terms containing a product of three T^i_j has led to the condition $q = u^2$ and to the following values of the parameters:

$$\lambda = -u^{-1} \qquad \mu = -u \qquad \xi = -u^{-2}. \tag{15}$$

Result 2. If the space of invariant forms ξ is supposed to be of dimension three at most, the only possibility to construct an invariant differential calculus on the quantum space defined by (2) is to have $q = u^2$; the defining relations are then given by (12).

In conclusion, the condition $q = u^2$ is necessary to be able to set up a G-invariant differential calculus on the quantum space defined by (2). In particular, the so-called q-oscillator which corresponds to $q = u^{-2}$ is excluded.

5. Quantum groups

5.1. A quantum group with seven generators

A unique set of relations $(R_{xx}, R_{\xi,\xi})$ has been obtained. They are given by (2, 12) with $q = u^2$ and define the x and ξ spaces respectively. The constraints of invariance by homomorphism δ and the consistency of the computations lead to the following relations

R_{TT} between the elements T^i_j:

$$
\begin{array}{llll}
T^1_1T^1_2 = u^{-2}T^1_2T^1_1 & T^1_1T^1_3 = u^{-1}T^1_3T^1_1 & T^1_1T^2_2 = T^2_2T^1_1 & T^1_1T^2_1 = u^2T^2_1T^1_1 \\
T^1_1T^2_3 = uT^2_3T^1_1 & T^1_1T^3_3 = T^3_3T^1_1 & T^1_2T^1_3 = uT^1_3T^1_2 & T^1_2T^2_2 = u^2T^2_2T^1_2 \\
T^1_2T^2_1 = u^4T^2_1T^1_2 & T^1_2T^2_3 = u^3T^2_3T^1_2 & T^1_2T^3_3 = u^2T^3_3T^1_2 & T^1_3T^2_2 = uT^2_2T^1_3 \\
T^1_3T^2_1 = u^3T^2_1T^1_3 & T^1_3T^3_3 = uT^3_3T^1_3 & T^2_2T^2_1 = u^2T^2_1T^2_2 & T^2_2T^2_3 = uT^2_3T^2_2 \\
T^2_2T^3_3 = T^3_3T^2_2 & T^2_1T^2_3 = u^{-1}T^2_3T^2_1 & T^2_1T^3_3 = u^{-2}T^3_3T^2_1 & T^2_3T^3_3 = u^{-1}T^3_3T^2_3
\end{array}
$$

$$T^1_3T^2_3 - u^2T^2_3T^1_3 + s(T^1_1T^2_2 - u^2T^2_1T^1_2 - (T^3_3)^2) = 0. \tag{16}$$

The inverse matrix can now be completely determined and is given by

$$T^{-1} = \begin{pmatrix} T^2_2T^3_3 & -u^2T^1_2T^3_3 & T^1_2T^2_3 - uT^1_3T^2_2 \\ -u^{-2}T^2_1T^3_3 & T^1_1T^3_3 & -u^{-2}T^1_1T^2_3 + u^{-3}T^1_3T^2_1 \\ 0 & 0 & T^1_1T^2_2 - u^{-2}T^1_2T^2_1 \end{pmatrix} \times D^{-1} \tag{17}$$

where the determinant D can be written as

$$D \equiv (T^1_1T^2_2 - u^{-2}T^1_2T^2_1)T^3_3. \tag{18}$$

Remark that the determinant is not central and its inverse D^{-1} must be added to the set of generators of the algebra. Its commutation relations $R_{TD^{-1}}$ are easily deduced from those of D and read:

$$
\begin{array}{lll}
D^{-1}T^1_1 = T^1_1D^{-1} & u^{-6}D^{-1}T^1_2 = T^1_2D^{-1} & u^{-3}D^{-1}T^1_3 = T^1_3D^{-1} \\
D^{-1}T^2_1 = u^{-6}T^2_1D^{-1} & D^{-1}T^2_2 = T^2_2D^{-1} & D^{-1}T^2_3 = u^{-3}T^2_3D^{-1}
\end{array}
$$

$$D^{-1}T^3_3 = T^3_3D^{-1}. \tag{19}$$

With these definitions, it may be verified that $H_8 \equiv C\langle T, D^{-1}\rangle / R_{TT} \cup R_{TD^{-1}}$ is a Hopf algebra with co-product Δ, co-unit ϵ and antipode S defined by

$$\Delta(T) \equiv T \otimes T \qquad \Delta(D^{-1}) \equiv D^{-1} \otimes D^{-1} \tag{20}$$

$$\epsilon(T, D^{-1}) \equiv (I, 1) \qquad S(T) \equiv T^{-1} \qquad S(D) \equiv D^{-1}. \tag{21}$$

Now we can apply the usual method [3, 4] to obtain the quadratic relations between the variables, the differentials and the derivatives [7]. They are given by

$$x^k\xi^l = \widehat{R}^{kl}_{mn}\xi^m x^n \tag{22}$$

$$\partial_k\xi^l = \widehat{R}^{-1lm}_{kn}\xi^n\partial_m \tag{23}$$

$$\partial_l x^k = \delta^k_l + \widehat{R}^{km}_{ln}x^n\partial_m. \tag{24}$$

5.2. A quantum group with nine generators

Once the matrix $\widehat{R}$ has been explicitly computed, it is possible to introduce a new quantum matrix t with nine elements satisfying the relations R_{tt} deduced from (9)

$$\widehat{R}^{ji}_{kl}t^k_m t^l_n = t^j_l t^i_k \widehat{R}^{lk}_{mn}. \tag{25}$$

The computation of the inverse t^{-1} yields

$$\begin{pmatrix} t^2_2t^3_3 - ut^2_3t^3_2 & -u^2t^1_2t^3_3 + u^3t^1_3t^3_2 & t^1_2t^2_3 - ut^1_3t^2_2 \\ -u^{-2}t^2_1t^3_3 + u^{-3}t^2_3t^3_1 & t^1_1t^3_3 - u^{-1}t^1_3t^3_1 & -u^{-2}t^1_1t^2_3 + u^{-3}t^1_3t^2_1 \\ t^2_1t^3_2 - u^{-2}t^2_2t^3_1 & -u^2t^1_1t^3_2 + t^1_2t^3_1 & t^1_1t^2_2 - u^{-2}t^1_2t^2_1 \end{pmatrix} d^{-1} \tag{26}$$

with the determinant d of t equal to

$$d = t^1_1t^2_2t^3_3 + t^1_3t^2_1t^3_2 + u^{-3}t^1_2t^2_3t^3_1 - u^{-1}t^1_1t^2_3t^3_2 - u^{-2}t^1_2t^2_1t^3_3 - u^{-2}t^1_3t^2_2t^3_1.$$

It can be verified that d is not a central element of $C\langle t\rangle$ and therefore must be added to this algebra. The commutation relations $R_{td^{-1}}$ of d^{-1} with the generators t^i_j are

$$\begin{array}{lll} t^1_1 d^{-1} = d^{-1} t^1_1 & t^1_2 d^{-1} = u^{-6} d^{-1} t^1_2 & t^1_3 d^{-1} = u^{-3} d^{-1} t^1_3 \\ t^2_2 d^{-1} = d^{-1} t^2_2 & t^2_1 d^{-1} = u^{4} d^{-1} t^2_1 & t^2_3 d^{-1} = u^{-3} d^{-1} t^2_3 \\ t^3_1 d^{-1} = u^{3} d^{-1} t^3_1 & t^3_2 d^{-1} = u^{-3} d^{-1} t^3_2 & t^3_3 d^{-1} = d^{-1} t^3_3. \end{array} \qquad (27)$$

In this manner, $H_{10} \equiv C\langle t, d^{-1}\rangle / R_{tt} \cup R_{td^{-1}}$ is endowed with a structure of Hopf algebra.

Thus two Hopf algebras, H_8 and H_{10}, have been constructed. Both preserve the same differential calculus on the deformed oscillator algebra defined by R_{xx} with $q = u^2$. In addition, the construction ensures that H_{10} contains $H_8 = C\langle T, D^{-1}\rangle / R_{TT} \cup R_{TD^{-1}}$ as a Hopf subalgebra.

6. Conclusion

We have been able to deform simultaneously the Weyl–Heisenberg algebra and its group of invariance (a subgroup of $GL(3)$). In addition, an invariant differential calculus has been set up on the resulting quantum space. However, it must be stressed that the whole construction cannot be carried out for arbitrary values of the deformation parameters and that the final result depends only on one complex number u.

The constraint on the values of the parameters can be removed when the requirement of invariance by a seven-generator quantum group is lifted. A purely algebraic approach [4] can be developed and the commutation relations are then shown to be invariant by a quantum matrix belonging to a three-parameter deformation of $GL(3)$ [8, 9].

The construction performed in this paper has yielded two quantum groups and their associated Hopf algebras H_{10}, H_8, which have ten and eight generators, respectively. These algebras are original deformations of $GL(3)$ and of its subgroup G. They are different from $GL_q(3)$ and other quantum groups proposed in [5] since they correspond to the conservation of different quadratic forms. Moreover, the smaller one H_8 is embedded in H_{10} as a true Hopf subalgebra.

References

[1] Woronowicz S L 1989 Differential calculus on compact matrix pseudogroups (quantum groups) *Commun. Math. Phys.* **122** 125–70

[2] Bernard D 1990 Quantum Lie algebras and differential calculus on quantum groups *Prog. Theor. Phys. Suppl.* **102** 49–66

[3] Wess J and Zumino B 1990 Covariant differential calculus on the quantum hyperplane *Nucl. Phys. (Proc. Suppl.)* B **18** 301–13

[4] Zumino G 1992 Differential calculus on quantum spaces and quantum groups *Group Theoretical Methods in Physics* ed M A del Olmo, M Santander and J Mateos Guilarte (CIEMAT, Spain: Anales de Física Monografias) pp 41–59

[5] Reshetikhin N Yu, Takhtadzhyan L A and Faddeev L D 1990 Quantization of Lie groups and Lie algebras *Leningrad Math. J.* **1** 193–225

[6] Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)

[7] Bertrand J and Irac-Astaud M 1995 Invariant differential calculus on a deformation of the Weyl–Heisenberg algebra *Modern Group Theoretical Methods in Physics* (Dordrecht: Kluwer) pp 37–50

[8] Irac-Astaud M 1996 Differential calculus on a three-parameter oscillator algebra *Rev. Mod. Phys.* **8** 1083–90

[9] Irac-Astaud M 1996 A three-parameter deformation of the Weyl–Heisenberg algebra: differential calculus and invariance *Quantum Groups and Integrable Systems (Prague), Czech. J. Phys.* to be published